5.4 Open sets

5.4.A Definition: Let M be a metric space. We say that the subset G of M is an open subset of M (or simply, that G is open) if for every $x \in G$ there exists a number $r > 0$ such that the entire open ball $B[x;r]$ is contained in G.

5.4 B Theorem: In any metric space $\langle M, \rho \rangle$ both M and the empty sets ϕ are open sets.

Proof: If $x \in M$ then [by definition of $B[x;r]$ every open ball $B[x;r]$ is contained in M. Therefore M is open. The empty set ϕ is open simply because there are no x in ϕ and hence every $x \in \phi$ satisfies the condition of 5.4A.

5.4 C Theorem: Let $\mathcal{F}$ be any nonempty family of open subsets of a metric space M. Then $\bigcup_{G \in \mathcal{F}} G$ is also open subset of M.

Let $H = \bigcup_{G \in \mathcal{F}} G$ To prove H is open set.

We may assume that atleast one $G \in \mathcal{F}$ is not empty. (otherwise $H = \phi$ is open by 5.4B).

Choose any $x \in H$, then $x \in G$ for some $G \in \mathcal{F}$.

Since G is open there is some $B[x;r]$ with

$B[x;r] \subset G$ But $G \subset H \Rightarrow B[x;r] \subset H$

$\Rightarrow$ H is open

$\Rightarrow \bigcup_{G \in \mathcal{F}} G$ is open.

Theorem 5·4 D

If G_1 and G_2 are open subsets of the metric space M, then $G_1 \cap G_2$ is also open.

Proof we may assume that $G_1 \cap G_2 \neq \phi$.

If $x \in G_1 \cap G_2$

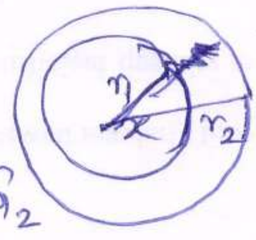
$\Rightarrow x \in G_1$ and $x \in G_2$

$x \in G_1 \Rightarrow$ since G_1 is open there exists $r_1 > 0$ such that $B[x; r_1] \subset G_1$

similarly $x \in G_2 \Rightarrow$ there exists $r_2 > 0$ such that The open ball $B[x; r_2] \subset G_2$

Thus if $r = \min\{r_1, r_2\}$

then $B[x; r]$ is contained in G_1 and G_2



$\Rightarrow B[x; r] \subset G_1 \cap G_2$

$\Rightarrow G_1 \cap G_2$ is open set.

Theorem:

Every open subset G of R^1 can be written $G = \cup I_n$ where $I_1, I_2, \ldots$ are a finite number of or countable number of open intervals which are mutually disjoint. (That is $I_m \cap I_n = \phi$ if $m \neq n$)

Proof

If $x \in G$, then there is an open interval (open ball) B containing x such that $B \subset G$.

Let I_x denote the largest open interval containing x such that $I_x \subset G$.

[I_x may be an unbounded interval, for example (a, ∞)]. Then $G = \bigcup_{x \in G} I_x$.

Now if $x \in G$, $y \in G$, then either $I_x = I_y$ or $I_x \cap I_y = \phi$. For if $I_x \neq I_y$ and $I_x \cap I_y \neq \phi$, then $I_x \cup I_y$ would be an open interval contained in G which is larger than I_x. This contradicts the definition of I_x. Finally, each I_x contains a rational number. Since disjoint intervals cannot contain the same rational and since there are only countabley many rationals, there cannot be uncountably many mutually disjoint intervals I_x.

Theorem 5.4 G Let $\langle M_1, \rho_1 \rangle$ and $\langle M_2, \rho_2 \rangle$ be metric spaces and let $f : M_1 \to M_2$. Then f is continuous on M_1 if and only if $f^{-1}(G)$ is open in M_1 when ever G is open in M_2.

(or)

(f is continuous if and only if $\cancel{f^{-1}(G)}$ The inverse image of every open set is open)

<u>Proof</u> Suppose first that f is continuous on M_1 we must to show that if G is open in M_2 then $f^{-1}(G)$ is open in M_1.

Thus, If $x \in f^{-1}(G)$, we must find an open ball $B[x;r]$ contained in $f^{-1}(G)$.

Now since $x \in f^{-1}(G) \Rightarrow f(x) \in G$, then $y = f(x) \in G$

Hence there is ~~let~~ an open ball $B[y;s]$ contained in G. (Since G is open in M_2)

$\therefore$ By the theorem 5.3C, $f^{-1}[B[y;s]]$ contains some $B[x;r]$. Hence $f^{-1}(G) \supset f^{-1}(B[y;s] \supset B[x;r$

Hence proved part(i)

conversely.

Now suppose $f^{-1}(G)$ is open in M_1 when ever G is open in M_2.

To show that f is continuous on M_1, it is sufficient to show that f is continuous at an arbitrary point $a \in M_1$.

Let $\mathcal{B} = B[f(a); \epsilon]$ be any ball about $f(a)$.

Then $\mathcal{B}$ is open in M_2 and so, by assumption $f^{-1}(\mathcal{B})$ is open in M_1. Since $a \in f^{-1}(\mathcal{B})$ and $f^{-1}(\mathcal{B})$ is open, there is an open ball $B[a;\delta]$ contained in $f^{-1}(\mathcal{B}]$. But then by (b) of 5.3C f is continuous at a.

This completes the proof.